On the algebraicity of GKZ-hypergeometric functions defined by a (hyper)cuboid

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Abstract

GKZ-hypergeometric functions are a very general extension of hypergeometric functions. This thesis contains a nearly-complete analysis of the algebraicity of the solutions to a GKZ-system (see definition 2.1) where α is rational and A is either a rectangle, a cuboid or a hypercuboid of arbitrary dimension. It turns out that there are very few cases in which the system has algebraic solutions: see section 4.4 for details.

Contents

1 Introduction

Hypergeometric functions have a long history: Gauss (and later Riemann and Kummer) already studied what is now known as the "classical" hypergeometric function, ${}_2F_1\left({\alpha\ \beta \atop \gamma}|z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n$, where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$ is the Pochhammer symbol. An important motivation for studying this function is that it is related to many well-known functions, including, but not limited to

$$\begin{aligned} (1+z)^a &= {}_2F_1\left({-a\ 1 \atop 1}|-z\right) \\ \frac{\arcsin(z)}{z} &= {}_2F_1\left({\frac{1}{2}\ \frac{1}{2} \atop \frac{3}{2}}|z^2\right) \\ \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}} &= {}_2F_1\left({\frac{1}{2}\ \frac{1}{2} \atop 1}|z\right) \end{aligned}$$

which means that results obtained for this hypergeometric function are potentially applicable in a wide variety of cases.

In 1873, Hermann Schwarz (probably best known for the Cauchy-Schwarz inequality) provided a list of all $(\alpha, \beta, \gamma) \in \mathbb{Q}^3$ for which the classical hypergeometric function is algebraic (in z). Quite a few people have extended this list to various generalizations of ${}_2F_1\left({\alpha\ \beta \atop \gamma}|z\right)$ in the last decades; my supervisor Frits Beukers, in cooperation with Gert Heckman, extended this list to general hypergeometric functions of one variable (i.e. ${}_pF_{p-1}\left({\alpha_1\ \alpha_2\ \ldots\ \alpha_p \atop \beta_1\ \beta_2\ \ldots\ \beta_{p-1}}|z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{n!(\beta_1)_n(\beta_2)_n\cdots(\beta_{p-1})_n} z^n$) in [BH89].

However, the classical hypergeometric function can be extended in other ways as well, and analogous lists have been created for many of these extensions. In the 1990's, Gel'fand, Kapranov and Zelevinsky proposed the very general class of A-hypergeometric functions (now commonly called "GKZ-hypergeometric functions" in their honour; see definition 2.1) Although this concept is rather abstract, it includes the above ${}_pF_{p-1}$ as well as many of the other extensions as special cases. Obviously, it would be tremendously useful to have at least a partial list of parameters for which this far more general function is algebraic. A research project to this end has been started under grant OND1331860 from the Netherlands Organization for Scientific Research (NWO).

This Bachelor's thesis is an attempt to provide, if not a building block, then at least a couple of examples for those undertaking this research. In particular, I have some hope that the idea behind theorem 3.6 may be more generally applicable, and I have some hope that the results obtained in this thesis may be a useful "reality check" for the lofty proofs likely to be required for the more-general theorems this project will require.

Additionally, with a simple linear transformation (which does not invalidate theorem 2.4), the Horn series G_3 can be transformed into the configuration $(r, n_1) = (2, 3)$ (in terms of definition 3.1). Thus, this thesis also proves the conjecture made by Beukers in [Beu07] that there are but a few choices of parameters such that the Horn series G_3 is algebraic (see section 4.4 for the final results, which imply this one; the proof consists of almost the entire thesis).

This thesis follows a rather simple plan of attack: we uses theorem 2.4 extensively to reduce this difficult analytic problem to a less-difficult combinatorial problem, and then repeatedly prove something conclusive for a wide range of parameters that were not handled by a previous argument. Thus, the progression from more to less general is a continuing theme in this thesis. However, in quite a few of the cases, the complexity is inherent, not accidental: for instance, a significant proportion of the possible values for parameters "missed" by proofs of non-algebraicity like corollary 3.8 and lemma 3.17 do actually lead to algebraic solutions.

2 Preliminary theory

By and large, this thesis will skip the more analytic parts and restrict itself to the combinatorial arguments made possible by theorem 2.4. However, we do need a couple of definitions.

Definition 2.1. Let $A = \{a_1, a_2, \ldots, a_N\}$ be a finite subset of $\mathbb{Z}^r$ such that the $\mathbb{Z}$-span of A is $\mathbb{Z}^r$ and such that there is a linear form h such that $h(a_1) = h(a_2) = \ldots = h(a_N) = 1$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r$.

Let $L = \{I = (I_1, I_2, \ldots, I_N) \in \mathbb{Z}^N | \sum_{i=1}^N I_i a_i = 0\}$. Let $v_1, \ldots, v_N$ be variables. For all $I \in L$ and all linear forms m, define

$$\Box_I = \prod_{I_i>0} \left(\frac{\partial}{\partial v_i}\right)^{I_i} - \prod_{I_i<0} \left(\frac{\partial}{\partial v_i}\right)^{-I_i}$$

$$Z_m = \left(\sum_{i=1}^N m(a_i) v_i \frac{\partial}{\partial v_i}\right) + m(\alpha)$$

Then the *GKZ-system associated to A and α* consists of the following equations:

$$\Box_I \phi = 0 \qquad \text{for each } I \in L$$
$$Z_m \phi = 0 \qquad \text{for each linear form } m$$

Again, we will restrict ourselves to combinatorial arguments, which will be far more elementary than the above definition suggests. We do frequently need the following definitions, but these are hardly complicated. Note that there cannot be more apex points than the number that is defined as "maximal" below, as a trivial consequence of theorem 3.6, so the name is at least somewhat justified.

Definition 2.2. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set of points in $\mathbb{R}^r$. Then the *positive real cone* spanned by A is the set $C(A) = \{\sum_{i=1}^n \lambda_i a_i | \forall i : \lambda_i \in \mathbb{R}_{\geq 0}\}$.

Definition 2.3. A point $p \in \mathbb{R}^r$ is an *apex point* if it is in $K(\alpha, A) = C(A) \cap (\alpha + \mathbb{Z}^r)$ and there is no $q \in K(\alpha, A) \setminus \{p\}$ such that $p - q \in C(A)$. If the number of apex points is equal to the *simplex volume*[1] of the convex hull of A, we call it *maximal.*

Finally, let us formulate the main theorem we will base our analysis on. The interested reader is referred to [Beu07] for any and all details: we will assume this as a given.

[1]The usual volume, corrected such that the fundamental simplex has volume 1 – that is, the simplex volume of an r-dimensional object is $r!$ times the normal volume.

Theorem 2.4. *Consider the GKZ-system associated to $A \subset \mathbb{Z}^r$ and $\alpha \in \mathbb{Q}^r$. Suppose that every element of $K(\alpha, A)$ can be written as a linear combination of the elements of A with non-negative integer coefficients (normality assumption). Furthermore, suppose that $\alpha + \mathbb{Z}^r$ has no points on the boundary of $C(A)$ (the GKZ-system is irreducible).*

Then the GKZ-system has a solution space consisting of algebraic functions exactly if the number of apex points in $K(k\alpha, A)$ is equal to the volume of the convex hull of A for all integers k such that $\gcd(k, N) = 1$, where N is the smallest integer such that $N\alpha \in \mathbb{Z}^r$.

Remark 2.5. Under the conditions of theorem 2.4, the GKZ-system has an algebraic solution exactly if the solution space consists of algebraic functions; that is, it either has no algebraic solutions, or every solution is algebraic.

3 The general case

For the rest of this thesis, we will use the following definitions and symbols:

Definition 3.1. Let r be an integer greater than 1 and let $n_1, n_2, \ldots, n_{r-1}$ be positive integers. Let $A = \{a \in \mathbb{Z}^r | \forall i \in \{1, 2, \ldots, r-1\} : 0 \leq a_i \leq n_i \wedge a_r = 1\}$ be an $r-1$-dimensional (hyper)cuboid. Finally, let $\alpha \in \mathbb{Q}^r$ be such that the GKZ-system of equations associated to A and α is irreducible.

The above definition appears fairly general, but we will prove that there are but a few specific values of r and α such that the associated system has a solution space consisting entirely of algebraic functions. We will start by finding bounds on the number and location of apex points, notably theorem 3.6; we will then utilize those bounds, and occasionally theorem 2.4 directly, to obtain our results.

The following definition and lemma, presented graphically in figure 3.1, will prove tremendously useful. Note that the conclusion, presented in section 4.4, is formulated using only the previous chapter and definitions 3.1 and 3.2, so a reader who has come this far should be able to understand it.

Definition 3.2. We will denote the *fractional part* of $x \in \mathbb{R}$ as $\{x\} = x - \lfloor x \rfloor$. For $x \in \mathbb{R}^r$, we define $\{x\}$ componentwise, i.e. $\{x\} = (\{x_1\}, \{x_2\}, \ldots, \{x_r\})$.

Lemma 3.3. *A point* $p \in \alpha + \mathbb{Z}^r$ *is an apex point exactly if* $\forall j \in \{1, 2, \ldots, r-1\} : 0 \leq p_j \leq n_j p_r$ *and* $\exists i \in \{1, 2, \ldots, r-1\} : n_i(p_r - 1) < \{p_i\}$.

Proof. Note that $C(A)$ is a cone spanned by the (hyper)cuboid A with its bottom at 0. Thus, a point $p \in \mathbb{Q}^r$ is in $C(A)$ exactly if $\forall j \in \{1, 2, \ldots, r-1\}$: $0 \leq p_j \leq n_j p_r$.

A point $p \in (\alpha + \mathbb{Z}^r) \cap C(A)$ is an apex point exactly if there is no $q \in ((\alpha + \mathbb{Z}^r) \cap C(A)) \setminus \{p\}$ such that $p - q \in C(A)$. Equivalently, a point $p \in (\alpha + \mathbb{Z}^r) \cap C(A)$ is an apex point if there is no $c = p - q \in (\mathbb{Z}^r \cap C(A)) \setminus \{0\}$ such that $p - c \in C(A)$.

Suppose that $p \in (\alpha + \mathbb{Z}^r) \cap C(A)$ is an apex point. For each $j \in \{1, 2, \ldots, r-1\}$, let $c_j = \min(\lfloor p_j \rfloor, n_j)$, and note $c = (c_1, c_2, \ldots, c_{r-1}, 1) \in (\mathbb{Z}^r \setminus \{0\}) \cap C(A)$. Note $p - c \notin C(A)$ by assumption. But $\forall j \in \{1, 2, \ldots, r-1\} : p_j - c_j \geq 0$,

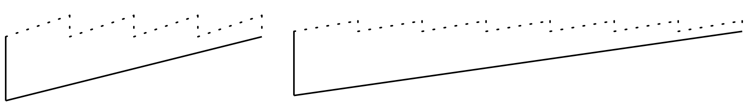

Figure 3.1: Per lemma 3.3, a point $p \in (\alpha + \mathbb{Z}^r) \cap C(A)$ is an apex point exactly if the projection onto the plane spanned by e_i and e_r is in the above area for some $i \in \{1, 2, \ldots, r-1\}$. (This presents the cases $n = 3$ and $n = 6$.)

so $\exists i \in \{1, 2, \ldots, r-1\} : p_i - c_i > n_i(p_r - c_r) = n_i(p_r - 1)$. If $c_i = n_i$, $p_i - n_i > n_i p_r - n_i$ so $p_i > n_i p_r$; but then $p \notin C(A)$. Therefore $c_i = \lfloor p_i \rfloor$ and $\{p_i\} = p_i - \lfloor p_i \rfloor > n_i(p_r - 1)$.

Conversely, suppose that p satisfies the above equations, and that $c \in (\mathbb{Z}^r \cap C(A)) \setminus \{0\}$. Choose an $i \in \{1, 2, \ldots, r-1\}$ such that $\{p_i\} > n_i(p_r - 1)$; then either $p_i - c_i < 0$ (and $p - c \notin C(A)$) or $\lfloor p_i \rfloor \geq c_i$, and therefore

$$\begin{aligned} p_i - c_i &= \{p_i\} + \lfloor p_i \rfloor - c_i \\ &> n_i(p_r - 1) + \lfloor p_i \rfloor - c_i \\ &= n_i(p_r - c_r) + n_i(c_r - 1) + \lfloor p_i \rfloor - c_i \\ &\geq n_i(p_r - c_r) \end{aligned}$$

which also implies $p - c \notin C(A)$. □

This description of apex points is much more convenient than definition 2.3. We will use it to obtain relations between apex points.

Remark 3.4. Let p and q be apex points. Then p and q are in $\alpha + \mathbb{Z}^r$, so $\{p\} = \{q\} = \{\alpha\}$.

Corollary 3.5. *Let p and q be apex points. Then $0 \leq p_r = q_r < 2$.*

Proof. Let p and q be apex points. There is an $i \in \{1, 2, \ldots, r-1\}$ such that $n_i(p_r - 1) < \{p_i\}$, so $p_r < \frac{\{p_i\}}{n_i} + 1 < 2$. So $0 \leq p_r < 2$, and likewise $0 \leq q_r < 2$.

Suppose $q_r \neq p_r$; we may assume, without loss of generality, that $q_r = p_r - 1$. Then $0 \leq q_i \leq n_i q_r = n_i(p_r - 1) < \{p_i\} = \{q_i\}$ and therefore $0 \leq q_i < \{q_i\}$, which is clearly not true. So $0 \leq p_r = q_r < 2$. □

3.1 The r-dimensional case (for $r > 3$)

We can now count the number of apex points in various configurations.

Theorem 3.6. *Suppose that $p \in \alpha + \mathbb{Z}^r$ is an apex point. Let $I = \{i \in \{1, 2, \ldots, r-1\} | n_i(p_r - 1) < \{p_i\}\}$, $I^c = \{1, 2, \ldots, r-1\} \setminus I$. Then there are no more than $\prod_{i \in I} n_i \prod_{j \in I^c} (\lfloor n_j p_r \rfloor + 1)$ apex points, and this bound is attained exactly if $\forall i \in I : n_i(p_r - 1) < \{p_i\} \leq n_i(p_r - 1) + 1$ and $\forall j \in I^c : \{p_j\} \leq \{n_j p_r\}$.*

Proof. By lemma 3.3, remark 3.4 and corollary 3.5, all apex points are of the form $\{p\} + c$, where $c \in \mathbb{Z}_{\geq 0}^{r-1} \times \{0\}$.

For each $j \in I^c$ we require $0 \leq \{p_j\} + c_j \leq n_j p_r$. So if we project all apex points onto the plane spanned by the standard basis vectors e_j and e_r, we obtain at most $\lfloor n_j p_r \rfloor + 1$ points; equivalently, there are at most $\lfloor n_j p_r \rfloor + 1$ integers c_j satisfying this inequality. Note that for each $c \in \mathbb{Z}$ such that $0 \leq \{p_j\} + c_j \leq n_j p_r$, there actually is an apex point that is projected to $\{p_j\} + c_j$, for instance $\{p\} + c_j e_j + \lfloor p_r \rfloor e_r$. There are exactly $\lfloor n_j p_r \rfloor + 1$ projected points/integers if and only if $\{p_j\} + \lfloor n_j p_r \rfloor \leq n_j p_r$, i.e. exactly if $\{p_j\} \leq \{n_j p_r\}$.

For each $i \in I$ we require $0 \leq \{p_i\} + c_i \leq n_i p_r < \{p_i\} + n_i$. So there are at most n_i integers satisfying this inequality (or n_i points in the image of the projection of the apex points, as above). This second bound is attained exactly if

$$\begin{aligned}
&\{p_i\} + n_i - 1 \leq n_i p_r < \{p_i\} + n_i \\
&\{p_i\} - 1 \leq n_i(p_r - 1) < \{p_i\} \\
&n_i(p_r - 1) < \{p_i\} \leq n_i(p_r - 1) + 1
\end{aligned}$$

Note that $\{p\}+c$ is an apex point if and only if it satisfies all the conditions imposed by lemma 3.3, i.e. if each $\{p_i\} + c_i$ satisfies the conditions imposed above. Therefore, the number of c – and hence the number of apex points – that satisfy all conditions is exactly the product of the number of c_j that satisfy their respective conditions, i.e. $\prod_{i\in I} n_i \prod_{j\in I^c}(\lfloor n_j p_r\rfloor + 1)$. ☐

The convex hull of A is obviously an $(r-1)$-dimensional orthotope. Recall our definition of the simplex volume (see definition 2.3), which states that the simplex volume is $n!$ times the normal volume. Therefore,

Remark 3.7. The simplex volume of the convex hull of A is $(r-1)!\prod_{j=1}^{r-1} n_j$.

Using the above remark and lemma, we can find a result that allows us to exclude most instances of A from further consideration.

Corollary 3.8. *If $r > 3$, there are no algebraic solutions.*

Proof. Recall corollary 3.5, which gives us that $p_r < 2$. Also note that $I \neq \emptyset$. Thus, the number of apex points is no larger than $\prod_{i\in I} n_i \prod_{j\in I^c}(\lfloor n_j p_r\rfloor+1) \leq 2^{r-2}\prod_{j=1}^{r-1} n_j$, which for $r > 3$ is smaller than the simplex volume of the convex hull, $(r-1)!\prod_{j=1}^{r-1} n_j$ (see remark 3.7). ☐

3.2 The two-dimensional case

The above Corollary 3.8 handles all high-dimensional cases, but leaves the cases where $r = 2$ or $r = 3$ unresolved. It will turn out that these are very similar. We will denote $\alpha = (\alpha_1, \alpha_r)$ respectively $\alpha = (\alpha_1, \alpha_2, \alpha_r)$, and $\{\alpha_i\} = \frac{\beta_i}{\gamma_i}$ for each applicable i, where β_i and γ_i are coprime nonnegative integers. We will also abbreviate n_1 to simply n.

We will start with some dimension-specific analysis, after which most of the theory will apply to each case.

Per theorem 3.6, the number of apex points is maximal exactly if $n(p_r-1) < \{p_1\} \leq n(p_r - 1) + 1$. Thus, the number of apex points is maximal exactly if there is an apex point in the area marked in figure 3.2. We can free ourselves from the dependence on p by only considering fractional parts:

Corollary 3.9. *The number of apex points is maximal exactly if*

$$\frac{\{\alpha_1\}}{n} > \{\alpha_r\} \quad or \quad \frac{\{\alpha_1\} + n - 1}{n} \leq \{\alpha_r\}$$

Proof. Per theorem 3.6, the number of apex points is maximal (i.e. n) exactly if $n(p_r-1) < \{p_1\} \leq n(p_r-1)+1$. Note that for any point p in $\alpha+\mathbb{Z}^r$, $\{p\} = \{\alpha\}$.

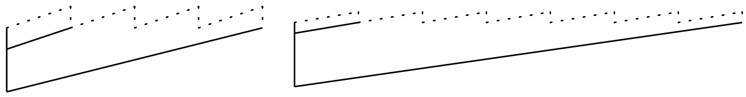

Figure 3.2: Per corollary 3.9, the number of apex points is maximal exactly if there is an apex point in the inner area. (Like figure 3.1, this presents the cases $n = 3$ and $n = 6$.)

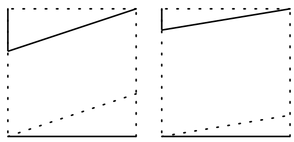

Figure 3.3: Per corollary 3.9, the number of apex points is maximal exactly if $\{\alpha\}$ is in this area. This presents the cases $n = 3$ and $n = 6$.

Now suppose that $\frac{\{\alpha_1\}}{n} > \{\alpha_r\}$. Let $p = (\{\alpha_1\}, \{\alpha_r\} + 1)$. Note $0 \leq p_1 < 1 \leq n(p_r - 1) + 1$, and $\{p_1\} = p_1 > n(p_r - 1)$, so p is indeed an apex point (lemma 3.3). Finally, note $n(p_r - 1) + 1 = n\{\alpha_r\} + 1 \geq 1 > \{\alpha_1\} = \{p_1\}$. So $n(p_r - 1) < \{p_1\} \leq n(p_r - 1) + 1$.

Likewise, suppose that $\frac{\{\alpha_1\}+n-1}{n} \leq \{\alpha_r\}$. Let $p = (\{\alpha_1\}, \{\alpha_r\})$. Note $0 \leq p_1 \leq p_r$, and $\{p_1\} \geq 0 > n(p_r - 1)$; additionally, $\{p_1\} \leq n(p_r - 1) + 1$. Therefore, $n(p_r - 1) < \{p_1\} \leq n(p_r - 1) + 1$.

Finally, assume that we have a p such that $n(p_r - 1) < \{p_1\} \leq n(p_r - 1) + 1$. Note $0 \leq p_r < 2$ (corollary 3.5). If $p_r \geq 1$, $n\{p_r\} = n(p_r - 1) < \{p_1\}$, so $\frac{\{p_1\}}{n} > \{p_r\}$. Otherwise, $\{p_1\} \leq n(p_r - 1) + 1 = n(\{p_r\} - 1) + 1$, so $\frac{\{p_1\}+n-1}{n} \leq \{p_r\}$. $\square$

The following statement is pretty obvious, but nonetheless very useful (cf. remark 3.14):

Remark 3.10. The number of apex points is not maximal if $\frac{1}{n} \leq \{\alpha_r\} < \frac{n-1}{n}$.

3.3 The three-dimensional case

Let us now turn our attention to the three-dimensional case. As in the two-dimensional case, theorem 3.6 can be greatly simplified.

Corollary 3.11. *The number of apex points is maximal exactly if there is an $i \in \{1, 2\}$, $j = 3 - i$, such that $1 + \frac{\{\alpha_j\}-1}{n_j} \leq \{\alpha_r\} < \frac{\{\alpha_i\}}{n_i}$.*

Proof. Note that there is always at least one apex point; so let p be an arbitrary apex point. Note $\{p\} = \{\alpha\}$. The number of apex points is maximal, i.e. equal to $2n_1n_2$, exactly if the bound $\prod_{i \in I} n_i \prod_{j \in I^c}(\lfloor n_j p_r \rfloor + 1)$ from theorem 3.6 equals $2n_1n_2$ and is attained (note that it cannot be larger, as $p_r < \frac{\{p_i\}}{n_i} + 1 < 2$).

Note that p is an apex point, so the conditions in lemma 3.3 hold. This leads to equations (3.1a), (3.1b) and (3.1c).

If $n_j(p_r-1) < \{p_j\}$, i.e. $j \in I$, $\prod_{i\in I} n_i \prod_{j\in I^c}(\lfloor n_j p_r\rfloor+1) = n_1n_2 < 2n_1n_2$, so there are no algebraic solutions unless $I=\{i\}$. So $j \in I^c$; this is equivalent to equation (3.1d). For the number of apex points to be maximal, it is necessary that $n_i(\lfloor n_j p_r\rfloor+1) = 2n_1n_2$, i.e. $\lfloor n_j p_r\rfloor = 2n_j-1$. This is the case exactly if $p_r \geq 2-\frac{1}{n_j}$, i.e. if equation (3.1e) holds.

Obviously, the bound mentioned in theorem 3.6 must actually be attained: this leads to equations (3.1f) and (3.1g). Therefore, the number of apex points is maximal (in the sense of theorem 2.4) exactly if all of the following equations hold:

$$
\begin{aligned}
0 \leq p_i &\leq n_i p_r && \text{(3.1a)}\\
0 \leq p_j &\leq n_j p_r && \text{(3.1b)}\\
n_i(p_r-1) &< \{p_i\} && \text{(3.1c)}\\
\{p_j\} &\leq n_j(p_r-1) && \text{(3.1d)}\\
2-\frac{1}{n_j} &\leq p_r && \text{(3.1e)}\\
n_i(p_r-1) &< \{p_i\} \leq n_i(p_r-1)+1 && \text{(3.1f)}\\
\{p_j\} &\leq \{n_j p_r\} && \text{(3.1g)}
\end{aligned}
$$

At first glance, this is a fairly impressive array of equations. However, it can be greatly simplified. If we combine (3.1e) and (3.1c), we find $1 \leq 2-\frac{1}{n_j} \leq p_r < 1+\frac{\{p_i\}}{n_i} < 2$. So we can substitute $\{n_jp_r\} = n_jp_r - \lfloor n_jp_r\rfloor = n_jp_r-(2n_j-1) = 1-n_j(2-p_r)$ in equation (3.1g), obtaining

$$0 \leq \{p_j\} \leq \{n_jp_r\} = 1-n_j(2-p_r) \tag{3.2}$$

In particular, this equation implies $0 \leq 1-n_j(2-p_r)$, so $2-\frac{1}{n_j} \leq p_r$, which is equation (3.1e). Additionally, $\{p_j\} \leq 1-n_j(2-p_r) = 1-n_j+n_j(p_r-1) \leq n_j(p_r-1)$, which is equation (3.1d). Also note $n_i(p_r-1)+1 > n_i(1-\frac{1}{n_j})+1 \geq 1 > \{p_i\}$; thus, (3.1c) and (3.2) imply (3.1f).

Thus, the whole system of equations is equivalent to (3.1a), (3.1b), (3.1c) and (3.2). In other words, $p \in C(A)$ and $2+\frac{\{p_j\}-1}{n_j} \leq p_r < \frac{\{p_i\}}{n_i}+1$. Finally, note that $1 \leq p_r < 2$; so $\{\alpha_r\} = \{p_r\} = p_r-1$ which leads to the above inequality. □

This may appear somewhat dissimilar to corollary 3.9, but it actually isn't. First consider the following corollary:

Corollary 3.12. *If $n_1 \geq 2$ and $n_2 \geq 2$, there are no algebraic solutions.*

Proof. Suppose $n_1 \geq 2$ and $n_2 \geq 2$. Then the inequality in corollary 3.11 becomes

$$\frac{1}{2} < 1-\frac{1-\{\alpha_j\}}{n_j} \leq \{\alpha_r\} < \frac{\{\alpha_i\}}{n_i} < \frac{1}{2}$$

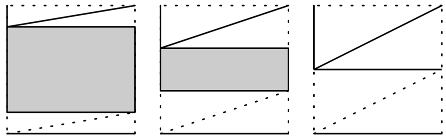

Figure 3.4: As noted in remark 3.14, the number of apex points is not maximal if $\{\alpha\}$ is in the inner area. This presents the cases $n = 6$, $n = 3$ and $n = 2$, illustrating why our proof doesn't yield any results in the last case.

which obviously has no solutions. □

Given the above, we may assume without loss of generality that $n_2 = 1$. Recall that we defined $n = n_1$. We find the following corollary, which is an obvious analogue of remark 3.10:

Corollary 3.13. *The number of apex points is not maximal if* $\frac{1}{n} \leq \{\alpha_r\} < \frac{n-1}{n}$.

Proof. Let us consider the inequality in corollary 3.11. Suppose $j = 1$. Obviously, $\frac{n-1}{n} \leq 1 - \frac{1-\{\alpha_1\}}{n} \leq \{\alpha_r\} < \{\alpha_2\}$, which obviously fails to hold if $\{\alpha_r\} < \frac{n-1}{n}$. Otherwise, $j = 2$, and $\{\alpha_1\} \leq \{\alpha_r\} < \frac{\{\alpha_2\}}{n} < \frac{1}{n}$, which obviously fails to hold if $\{\alpha_r\} \geq \frac{1}{n}$. □

3.4 Eliminating most possible values of α_r

Let us, for a moment, assume the conditions of theorem 2.4 are satisfied. We will use the intuition provided by figure 3.4: if $k\alpha_r \approx \frac{1}{2}$, it must satisfy remark 3.10 or corollary 3.13, as appropriate for the dimension.

Remark 3.14. The number of apex points is not maximal if $\frac{1}{n} \leq \{\alpha_r\} < \frac{n-1}{n}$.

Additionally, note that there is nothing to prove for $(r, n) = (2, 1)$: any choice of α leads to exactly one apex point, which is the maximum number. It is rather unfortunate, if perhaps unsurprising[1], that the following argument will not work for $(r, n) = (3, 1)$ or $n = 2$. So, in the following, we will always assume $n \geq 3$, and try to find an appropriate k for each α. We will leave troublesome values for α for later consideration.

The following lemmas take care of a lot of the fiddling that would otherwise be required to find appropriate k for use with remark 3.14 (and similar bounds.)

Lemma 3.15. *Let a, b and k be nonzero integers such that k is coprime with a. Then there is an integer l, coprime with ab, such that $l \equiv k \mod a$.*

Proof. Let c be the smallest positive integer such that $c|b$ and $\gcd(a, \frac{b}{c}) = 1$. There is an integer a' such that $aa' \equiv 1 \mod \frac{b}{c}$. Define $l = k - (k-1)aa'$. Then $l \equiv 1 \mod \frac{b}{c}$ and $l \equiv k \mod a$.

[1]This argument is mostly concerned with proving that there are no algebraic solutions: however, experimental results and (others') theoretical results suggest that there are a lot of configurations leading to algebraic solutions in those cases.

It remains to be shown that l is coprime with ab. Let p be an arbitrary prime dividing ab. Note that p divides either a or $\frac{b}{c}$. If p divides a, $l = k - (k-1)aa' \equiv k \mod p$. Since k is coprime with a, it is also coprime with p. If p divides $\frac{b}{c}$, l is coprime with p, since l is coprime with $a\frac{b}{c}$. □

Lemma 3.16. *Let $a \in \mathbb{R}_{<\frac{1}{2}}$ be a real number. Let b and c be coprime integers, and suppose $c \geq \max(3, \frac{d}{1-2a})$, where $d = 1$ if c is odd, $d = 2$ if $c \equiv 0 \mod 4$, and $d = 4$ if $c \equiv 2 \mod 4$. Then there is an integer k, coprime with c, such that $a \leq \{k\frac{b}{c}\} < \frac{1}{2}$, with equality exactly if $c = \frac{d}{1-2a}$.*

Proof. Since b and c are coprime, there is an integer b' such that $bb' \equiv 1 \mod c$. Let $k = b'\frac{c-d}{2}$; note $k \in \mathbb{Z}$. Obviously, b' is coprime with c, and therefore $\gcd(k,c) | \gcd(b',c)\gcd(\frac{c-d}{2}, c) = \gcd(\frac{c-d}{2}, c)$. If c is odd, $\gcd(\frac{c-d}{2}, c) | \gcd(c-1, c) = 1$; if $c \equiv 0 \mod 4$, $\gcd(\frac{c-d}{2}, c) = \gcd(\frac{c}{2} - 1, c) = 1$; and finally, if $c \equiv 2 \mod 4$, $\gcd(\frac{c-d}{2}, c) = \gcd(\frac{c}{2} - 2, c) = 1$. Thus, $\gcd(k, c) = 1$.

Additionally, $\{k\frac{b}{c}\} = \{b'\frac{c-d}{2}\frac{b}{c}\} = \{\frac{c-d}{2c}\}$. Note that $c \geq d$ for all $c \geq 3$, so $\{k\frac{b}{c}\} = \{\frac{c-d}{2c}\} = \frac{c-d}{2c} < \frac{1}{2}$. Finally, $a \leq \frac{c-d}{2c}$ exactly if $c \geq \frac{d}{1-2a}$, where we have equality on the left side if and only if we have equality on the right side. □

With these tools in hand, we can easily restrict the range of parameters that may result in algebraic solutions.

Lemma 3.17. *Suppose that $(\gamma_r, n) \notin \{(4,3), (6,3), (6,4), (6,5), (10,3)\}$. Then there are no algebraic solutions.*

Proof. We simply combine all statements above. Per remark 3.14, if $\frac{1}{n} \leq \{\alpha_r\} < \frac{n-1}{n}$, there are no algebraic solutions. Per lemma 3.16, we can find a k coprime with γ_r such that $\frac{1}{n} \leq \{k\alpha_r\} < \frac{1}{2} \leq \frac{n-1}{n}$, provided $\gamma_r \geq \max(3, \frac{dn}{n-2})$, where d is as defined in the lemma. This turns out to be the case except if $\gamma_r = 2$ (but in that case, simply consider remark 3.14 again) or $(\gamma_r, n) \in \{(4,3), (6,3), (6,4), (6,5), (10,3)\}$.

Per lemma 3.15, there is an integer l, coprime with $\prod_{i=1}^{r} \gamma_i$, such that $\frac{1}{n} \leq \{l\alpha_r\} < \frac{n-1}{n}$. □

4 Exceptions

We will now consider the cases left unresolved by lemma 3.17, i.e. $(\gamma_r, n) \in \{(4,3),(6,3),(6,4),(6,5),(10,3)\}$. This is only a handful, but they do require special attention. Let us first consider some corner cases: if $\{\alpha_1\} = 0$, or if $r = 3$ and $\{\alpha_2\} = 0$, there are no algebraic solutions. This is immediately obvious when considering the following remark and remark 3.14.

Remark 4.1. All solutions are algebraic for α if and only if all solutions are algebraic for $-\alpha$.

Proof. All solutions are algebraic for α if and only if the number of apex points is maximal for each $\{k\alpha\}$, where k is coprime with $\gamma_1\gamma_2\gamma_r$. Thus, all solutions are algebraic for α if and only if the number of apex points is maximal for each $\{-k\alpha\}$, where k is coprime with $\gamma_1\gamma_2\gamma_r$; that is, exactly if all solutions are algebraic for $-\alpha$. □

Now let us turn our attention to the special cases.

4.1 Results for $(\gamma_r, n) \in \{(4,3),(6,4),(6,5)\}$

If $\alpha_1 \neq 0$ and $(\gamma_r, n) \in \{(4,3),(6,4),(6,5)\}$, we can re-use the argument presented in section 3.4 to restrict γ_1 as well as γ_r. (It would be very convenient if this argument worked for the other two cases as well; unfortunately, I am not aware of any bounds on γ_1 that can easily be used with lemma 3.16.)

Lemma 4.2. *Let $(\gamma_r, n) \in \{(4,3),(6,4),(6,5)\}$. Then all solutions are algebraic exactly if $r = 2$ and $(\{\alpha_1\},\{\alpha_r\},n) \in \{(\frac{1}{6},\frac{5}{6},4),(\frac{5}{6},\frac{1}{6},4)\}$.*

Proof. For $r = 2$, the number of apex points for $\{k\alpha\}$ is not maximal exactly if $n(\{k\alpha_r\}-1)+1 < \{k\alpha_1\} \leq n\{k\alpha_r\}$, which is equivalent to either $\{k\alpha_1\} \leq \frac{n}{\gamma_r}$ or $\{k\alpha_1\} > \frac{\gamma_r - n}{\gamma_r}$.

For $r = 3$, the number of apex points for $\{k\alpha\}$ is not maximal exactly if $\forall i \in \{1,2\}, j = 3-i : 1 + \frac{\{k\alpha_j\}-1}{n_j} > \{k\alpha_r\} \vee \{k\alpha_r\} \geq \frac{\{k\alpha_i\}}{n_i}$, which is at least the case if $1 + \frac{\{k\alpha_1\}-1}{n} \leq \{k\alpha_r\} < \frac{\{k\alpha_i\}}{n}$. The last part is equivalent to $n\{k\alpha_r\} < \{k\alpha_1\} \leq 1 + n(\{k\alpha_r\}-1)$; considering the cases $\{k\alpha_r\} = \frac{1}{\gamma_r}$ and $\{k\alpha_r\} = \frac{\gamma_r - 1}{\gamma_r}$, as above, leads us to conclude that there are no algebraic solutions if $\frac{\gamma_r - n}{\gamma_r} < \{\alpha_1\} \leq \frac{n}{\gamma_r}$.

Provided $\gamma_1 > \frac{4\gamma_r}{2n-\gamma_r}$, we can now apply lemmas 3.16 and 3.15 to obtain an integer l, coprime with $\prod_{i=1}^{r}\gamma_i$, such that $\frac{\gamma_r - n}{\gamma_r} < \{k\alpha_1\} < \frac{1}{2} < \frac{n}{\gamma_r}$.

At this point, a computer can be used to solve the exceptions (i.e. the case $\gamma_1 \leq \frac{4\gamma_r}{2n-\gamma_r}$.) Appendix A contains the code and output. We will only mention the end result here: all solutions are algebraic if $r = 2$ and

$(\{\alpha_1\}, \{\alpha_r\}, n) \in \{(\frac{1}{6}, \frac{5}{6}, 4), (\frac{5}{6}, \frac{1}{6}, 4)\}$, and if $r = 2$ there are no other algebraic solutions.

If $r = 3$, there are no algebraic solutions unless $(\{\alpha_1\}, \{\alpha_r\}, n) \in \{(\frac{1}{6}, \frac{5}{6}, 4), (\frac{5}{6}, \frac{1}{6}, 4)\}$, as above (note that the requirements of corollary 3.11 include those of corollary 3.9; therefore, if $(\alpha_1, \alpha_2, \alpha_r)$ leads to algebraic solutions, so does (α_1, α_r)). We can simply fill in both possible values for (α_1, α_r) into the inequality in corollary 3.11 to find that (depending on k) we require either $\{k\alpha_2\} \le \frac{1}{6} < \frac{5}{24}$ or $\frac{19}{24} \le \frac{5}{6} < \{\alpha_2\}$; so in any case, if $\frac{1}{6} < \{k\alpha_2\} \le \frac{5}{6}$, there are no algebraic solutions. If we again apply the lemmas 3.16 and 3.15, we find that there are no algebraic solutions if $\gamma_2 > 6$. However, for any α_2, it's clearly possible to find a k such that $\{k\alpha_2\} > \frac{1}{2}$. (We remarked that there are no algebraic solutions if $\alpha_2 = 0$ back in section 3.4.) However, it's equally clear that we are not going to find any k such that $\{k\alpha_2\} > \frac{5}{6}$; therefore, there are no algebraic solutions if $r = 3$. □

4.2 Results for $(\gamma_r, n) = (6, 3)$

The last section does not handle $(\gamma_r, n) \in \{(6, 3), (10, 3)\}$. We will consider $(\gamma_r, n) = (6, 3)$ now, and handle $(\gamma_r, n) = (10, 3)$ in the next section (using essentially the same proof). This requires the following elementary lemma.

Lemma 4.3. *For all nonzero integers a, b, m and n such that* $\gcd(m, n) | a - b$, *there is an integer x such that $x \equiv a \mod m$ and $x \equiv b \mod n$.*

Proof. Let c be the integer defined by $a - b = c \gcd(m, n)$. Using the Euclidean algorithm, we can find integers d and e such that $\gcd(m, n) = dm + en$. Then $x = a - cdm = b + cen$ is the desired integer. □

The attentive reader may have noted the similarities between this lemma and lemma 3.15; the following proof likewise uses many of the same ideas as the proof of lemma 3.16.

Recall that we may still assume $\{\alpha_1\} \ne 0$ (see the previous section for details). By remark 4.1, it suffices to consider α where $\{\alpha_r\} = \frac{1}{6}$. Note that if $\{k\alpha_r\} = \frac{1}{6}$, remark 3.14 implies that the number of apex points is not maximal if $\{k\alpha_1\} \le \frac{1}{2}$.

Suppose $\gcd(\gamma_1, \gamma_r) \in \{1, 2\}$. Let β_1' be an arbitrary integer such that $\beta_1\beta_1' \equiv 1 \mod \gamma_1$. By lemma 4.3, above, there is an integer k such that

$$\begin{aligned} k &\equiv \beta_1' \mod \gamma_1 \\ k &\equiv 1 \mod \gamma_r \end{aligned}$$

since either $\beta_1' - 1 \equiv 0 \mod 1$ or $\beta_1' - 1 \equiv 1 - 1 \equiv 0 \mod 2$. Note $\gcd(k, \gamma_1\gamma_r) | \gcd(\beta_1', \gamma_1) \gcd(1, \gamma_r) = 1$ and $\{k\alpha\} = (\{\frac{1}{\gamma_1}\}, \frac{1}{6})$. Since $\{\frac{1}{\gamma_1}\} \le \frac{1}{2}$ for all γ_1, there are no algebraic solutions in either case.

Now suppose that $\gcd(\gamma_1, \gamma_r) \in \{3, 6\}$. Let β_1' be as above, let b and c be the largest integers such that $2^b 3^c | \gamma_1$, and choose $d \in \{1, 2, 3, 4, 6\}$ such that $\beta_1'(\frac{\gamma_1}{2^b 3^c} + d) \equiv 1 \mod \gcd(\gamma_1, \gamma_r)$. (It is not necessary to allow $d = 5$, as

$\pm1(\pm1+5) \not\equiv 1 \mod 6$ for any choice of signs.) By lemma 4.3, there is an integer k such that

$$k \equiv \beta_1'(\frac{\gamma_1}{2^b 3^c} + d) \mod \gamma_1$$
$$k \equiv 1 \mod \gamma_r$$

as $\beta_1'(\frac{\gamma_1}{2^b 3^c} + d) - 1 \equiv 1 - 1 \equiv 0 \mod \gcd(\gamma_1, \gamma_r)$.

Note $\gcd(k, \gamma_1\gamma_r) | \gcd(\beta_1', \gamma_1)\gcd(\frac{\gamma_1}{2^b 3^c} + d, \gamma_1)\gcd(1, \gamma_r) = \gcd(\frac{\gamma_1}{2^b 3^c} + d, \gamma_1)$. Let us suppose that there is some prime p dividing both $\frac{\gamma_1}{2^b 3^c} + d$ and γ_1. Clearly, either $p \in \{2, 3\}$ or $p|d$, so $p \in \{2, 3\}$. However, $\frac{\gamma_1}{2^b 3^c} + d \equiv \beta_1' \mod \gcd(\gamma_1, \gamma_r)$, so $\frac{\gamma_1}{2^b 3^c} + d$ is coprime with $\gcd(\gamma_1, \gamma_r)$. Note that $p|\gamma_1$ if and only if $p| \gcd(\gamma_1, \gamma_r)$, which implies $p \nmid \frac{\gamma_1}{2^b 3^c} + d$. Therefore, $\gcd(k, \gamma_1\gamma_r) = 1$.

Furthermore, $\{k\alpha\} = (\{\beta_1'(\frac{\gamma_1}{2^b 3^c} + d)\frac{\beta_1}{\gamma_1}\}, \frac{1}{6}) = (\{\frac{1}{2^b 3^c} + \frac{d}{\gamma_1}\}, \frac{1}{6})$. There are no algebraic solutions if $\frac{1}{2^c 3^c} + \frac{d}{\gamma_1} \leq \frac{1}{2}$, which is the case at least when $\gamma_1 \geq 36$. (Note $\frac{1}{2^b 3^c} \leq \frac{1}{3}$ and $d \leq 6$.)

Once again, we use a computer to solve the problem, and refer the interested reader to appendix A. For $r = 2$, we conclude that all solutions are algebraic if $(\{\alpha_1\}, \{\alpha_r\}, n) \in \{(\frac{1}{6}, \frac{5}{6}, 3), (\frac{5}{6}, \frac{1}{6}, 3), (\frac{1}{3}, \frac{5}{6}, 3), (\frac{2}{3}, \frac{1}{6}, 3)\}$. Once again, we simply fill in all possible values for $(\{\alpha_1\}, \{\alpha_r\}, n)$ in the inequality presented in corollary 3.11, invoke lemmas 3.16 and 3.15, and find that there are no algebraic solutions if $\frac{1}{6} < \{\alpha_2\} \leq \frac{5}{6}$ (we require either $\frac{1}{6} < \{\alpha_2\}$ or $\{\alpha_2\} \leq \frac{5}{6}$ for algebraicity), which is the case if $\gamma_2 > 6$. As before, for $r = 3$, we require that (α_1, α_r) leads to algebraic solutions. Furthermore, if $\gamma_1 = 6$, corollary 3.11 requires either $\frac{13}{18} \leq \frac{5}{6} < \{\alpha_2\}$ or $\{\alpha_2\} \leq \frac{1}{6} < \frac{5}{18}$; if $\gamma_1 = 3$, corollary 3.11 requires either $\frac{7}{9} \leq \frac{5}{6} < \{\alpha_2\}$ or $\{\alpha_2\} \leq \frac{1}{6} < \frac{2}{9}$, i.e. exactly the same. In either case, we can use the same argument as in the last section: by lemma 3.16 and 3.15, there are no algebraic solution unless $\gamma_2 \leq 6$; and if $\gamma_2 \leq 6$, we can always find a k such that $\{k\alpha_2\} > \frac{1}{2}$, but $\{k\alpha_2\} \leq \frac{5}{6}$ for any k.

In summary, all solutions are algebraic exactly if $r = 2$ and $(\{\alpha_1\}, \{\alpha_r\}, n) \in \{(\frac{1}{6}, \frac{5}{6}, 3), (\frac{5}{6}, \frac{1}{6}, 3), (\frac{1}{3}, \frac{5}{6}, 3), (\frac{2}{3}, \frac{1}{6}, 3)\}$.

4.3 Results for $(\gamma_r, n) = (10, 3)$

For $(\gamma_r, n) = (10, 3)$, we can use an argument that is extremely similar to the argument used for $(\gamma_r, n) = (6, 3)$, above.

By the same remark 4.1, we need only consider the cases where $\gamma_r \in \{\frac{1}{10}, \frac{3}{10}\}$ and $\alpha_1 \neq 0$. Analogously to the previous case, remark 3.14 tells us that the number of apex points is not maximal if $\{k\alpha_r\} = \{\alpha_r\}$ and $\{k\alpha_1\} \leq \frac{3}{10} \leq n\{k\alpha_r\} = 3\{\alpha_r\}$.

Suppose, once again, that $\gcd(\gamma_1, \gamma_r) \in \{1, 2\}$. Let β_1' be an arbitrary integer such that $\beta_1\beta_1' \equiv 1 \mod \gamma_1$. Once again, $\beta_1' - 1 \equiv 0 \mod \gcd(\gamma_1, \gamma_r)$, so we invoke lemma 4.3 and find that there is an integer k such that

$$k \equiv \beta_1' \mod \gamma_1$$
$$k \equiv 1 \mod \gamma_r$$

Note $\gcd(k, \gamma_1\gamma_r) | \gcd(\beta_1', \gamma_1) \gcd(1, \gamma_r) = 1$, and $\{k\alpha\} = (\{\frac{1}{\gamma_1}\}, \{\alpha_r\})$. Note that for $\gamma_1 \neq 2$, $\{\frac{1}{\gamma_1}\} \leq \frac{3}{10}$; we will consider the case $\gamma_1 = 2$ below.

Now suppose $\gcd(\gamma_1, \gamma_r) \in \{5, 10\}$. Let β_1' be as above; let b and c be the largest integers such that $2^b 5^c | \gamma_1$, and choose $d \in \{1, 2, 4, 5, 8, 10, 16\}$ such that $\beta_1'(\frac{\gamma_1}{2^b 5^c} + d) \equiv 1 \mod \gcd(\gamma_1, \gamma_r)$. (The gentle reader is invited to create a table of all pairs of numbers coprime with either 5 or 10 to see that these choices suffice.) Then there is an integer k such that

$$k \equiv \beta_1'(\frac{\gamma_1}{2^b 3^c} + d) \mod \gamma_1$$
$$k \equiv 1 \mod \gamma_r$$

as $\beta_1'(\frac{\gamma_1}{2^b 6^c} + d) - 1 \equiv 1 - 1 \equiv 0 \mod \gcd(\gamma_1, \gamma_r)$.

Note $\gcd(k, \gamma_1\gamma_r) | \gcd(\frac{\gamma_1}{2^b 5^c} + d, \gamma_1) = 1$. After all, suppose there is a prime p that divides both $\frac{\gamma_1}{2^b 5^c} + d$ and γ_1. Then either $p \in \{2, 5\}$ or $p | d$, so $p \in \{2, 5\}$. Note that, as above, $p | \gamma_1$ exactly if $p | \gcd(\gamma_1, \gamma_r)$, which implies $p \nmid \frac{\gamma_1}{2^b 5^c} + d$.

Analogously to the previous case, we find $\{k\alpha\} = (\{\beta_1'(\frac{\gamma_1}{2^b 5^c} + d)\frac{\beta_1}{\gamma_1}\}, \{\alpha_r\}) = (\{\frac{1}{2^b 5^c} + \frac{d}{\gamma_1}\}, \{\alpha_r\})$. Note $\frac{1}{2^b 5^c} \leq \frac{1}{5}$ and $d \leq 16$, and $\{\frac{1}{2^b 5^c} + \frac{d}{\gamma_1}\} \leq \frac{3}{10} \leq 3\{\alpha_r\} = n\{k\alpha_r\}$ at least if $\gamma_1 \geq 160$. Computer results (see appendix A) tell us that there are no algebraic solutions in any of these cases, including the case $\gamma_1 = 2$ noted above. (Note that the solutions for $(\alpha_1, \alpha_2, \alpha_r)$ cannot be algebraic unless the solutions for (α_1, α_r) are, so this conclusion is true for $r = 2$ as well as for $r = 3$.)

4.4 Conclusion

Collecting all our previous results, we can conclude the following[1]

Conclusion. *Suppose the conditions of theorem 2.4 are satisfied, that is, suppose that the GKZ-system is irreducible and that the normality assumption holds. If $\alpha_r \in \mathbb{Z}$, all solutions are algebraic. Otherwise, suppose that $(r, n_1, n_2, \ldots, n_{r-1}) \notin \{(2,2), (3,1,1), (3,1,2), (3,2,1)\}$. Then all solutions are algebraic exactly if $r = 2$ and $(\{\alpha_1\}, \{\alpha_2\}, n_1) \in \{(\{\alpha_1\}, \{\alpha_2\}, 1), (\frac{1}{3}, \frac{5}{6}, 3), (\frac{2}{3}, \frac{1}{6}, 3), (\frac{1}{6}, \frac{5}{6}, 3), (\frac{5}{6}, \frac{1}{6}, 3), (\frac{1}{6}, \frac{5}{6}, 4), (\frac{5}{6}, \frac{1}{6}, 4)\}$ (where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x).*

[1] The proof consists of the entire thesis, but to understand the conclusion one need only read chapter 2 (very short) and definitions 3.1 and 3.2.

A Computer code and output

A significant part of chapter 4 is concerned solely with finding bounds on the tuples (γ_1, γ_r, n) for which there may be an algebraic solution. This is highly successful: the theoretical arguments presented in that chapter give a definite result in all but a very small number of cases. Of course, a complete analysis also has to check for algebraicity in the (about 6000) cases that are not covered by these arguments. Note that this program only handles $r = 2$; the theoretical arguments presented in the chapter do a good job of handling $r = 3$ based on the results for $r = 2$ obtained in this appendix.

The actual calculations were performed using the following program, written for the Gambit-C 4.4.0 implementation of the Scheme programming language. It should work on pretty much any Scheme implementation[1], though.

The casual reader is invited to skip this whole appendix and/or run the program in verbose mode to spot-check the calculations (change the `(calculate ... #f)` calls at the end to `(calculate ... #t)` and expect a lot of output). The interested reader should at least be aware that the `(cond-expand)` forms only make sure that all proper SRFIs ("libraries") are available, by importing the reference implementation if necessary, and do very little interesting work.

```
(cond-expand
  (gambit
    ; Enable various (compile-time) optimizations
    (declare
      (block)
      (standard-bindings)
      (extended-bindings)
      (mostly-generic)))
  (else))

(cond-expand
  (srfi-1) ; ("List Library")
  (else
   ; Load reference implementation
   (load "lib/srfi-1-reference.scm")

   ; Required by reference implementation
   (define (check-arg pred val caller)
     (let lp ((val val))
       (if
         (pred val)
```

[1] It is written to require only R5RS, SRFI-0 ("Feature-based conditional expansion construct") and either SRFI-28 ("Basic Format Strings") or SRFI-6 ("Basic String Ports"), and the latter part can be disabled if the "verbose" mode is not required.

```
        val
        (lp (error "Bad argument" val pred caller)))))
```

; Replace (iota) with a version that works with just R5RS

```
(define (iota count . maybe-start+step)
  (let ((start (if (null? maybe-start+step)
                   0
                   (car maybe-start+step)))
        (step  (if (or (null? maybe-start+step)
                       (null? (cdr maybe-start+step)))
                   1
                   (cadr maybe-start+step))))
    (check-arg number? start iota)
    (check-arg number? step iota)
    (let loop ((n 0) (r '()))
      (if (= n count)
        (reverse r)
        (loop (+ n 1) (cons (+ start (* n step)) r))))))))
```

If SRFI-28 ("Basic Format Strings") is present, we can just use that. However, if that is not the case, we import the reference implementation, which requires SRFI-6 ("Basic String Ports") and SRFI-23 ("Error reporting mechanism"). If SRFI-23 is not present, we can just import the reference implementation of SRFI-23 first; however, SRFI-6 requires implementation support, so if the underlying system does not support it we sadly have to bail out. (As noted before, if verbose operation is not necessary, SRFI-28 – and thus SRFI-6 – is optional.)

```
(cond-expand
  (srfi-23
    (define has-srfi-23 #t))
  (else
    (define has-srfi-23 #f)))

(cond-expand
  (srfi-28)
  (else
    (cond-expand
      (srfi-6))
    (if (not has-srfi-23)
      (load "lib/srfi-23-reference.scm")
      (set! has-srfi-23 #t))
    (load "lib/srfi-28-reference.scm")))
```

This function shows the results of a `calculate` call in a readable format.

```
(cond-expand
  (gambit
    (define (show x)
      (if (or (not (list? x)) (not (null? x)))
        (pretty-print x))))
```

```
  (else
    (define (show x)
      (if (or (not (list? x)) (not (null? x)))
        (begin
          (write x)
          (newline))))))
```

With the (cond-expand) forms behind us, we begin by defining the fractional part ($\{x\}$).

```
(define (frp x)
  (- x (floor x)))
```

The following simple procedure returns all numbers in $\{1, 2, \ldots, n\}$ that are coprime with n. For instance, (coprimes 12) is (1 5 7 11). Anyone trying to do large-scale experiments with this code will soon find that this function is a performance bottleneck: contact the author for a memoizing version or implement one yourself (SRFI-69 may be a convenient starting point). However, the version below suffices for checking the ranges required for our proofs, and it's much easier to see that it's correct.

```
(define (coprimes n)
  (filter
    (lambda (m) (= (gcd m n) 1))
      (iota n 1)))
```

We define some helper functions to calculate the bound given in corollary 3.9. Note that the inequalities are given in a slightly different form here.

```
(define (lower-p1-bound k pr n)
  (* (frp (* k pr)) n))

(define (upper-p1-bound k pr n)
  (- (lower-p1-bound k pr n) n -1))

(define (in-bound k p1 pr n)
  (or (<= (frp (* k p1)) (upper-p1-bound k pr n))
      (>  (frp (* k p1)) (lower-p1-bound k pr n))))
```

In verbose mode, the program prints human-readable representations of the calculations being done. This representation is provided by the following function:

```
(define (equation-string k p1 pr n)
  (let* ((upper-bound (upper-p1-bound k pr n))
         (lower-bound (lower-p1-bound k pr n))
         (kp
           (if (= k 1)
             (format "{p_1} = ~a" (frp p1))
             (format "{~a p1} = ~a" k (frp (* k p1)))))
         (upper-bound-string (format "~a <= ~a" kp upper-bound))
```

```
        (lower-bound-string (format "~a < ~a" lower-bound kp)))
    (cond
      ((and (>= upper-bound 0)
            (<  lower-bound 1))
       (format "~a or ~a"
               upper-bound-string
               lower-bound-string))
      ((>= upper-bound 0)
       upper-bound-string)
      ((< lower-bound 1)
       lower-bound-string)
      (else
       (format "~a or ~a (No sol.)"
         upper-bound-string
         lower-bound-string)))))
```

We now come to the main calculation loop. This basically considers all possible values of β_r, γ_1, β_r and k and verifies corollary 3.9 for each combination. If verbose is true (`#t`), `calculate` produces human-readable output; in any case, it returns a list of (γ_1, γ_r, n) for which all solutions are algebraic.

```
(define (calculate gr n g1s verbose)
  (let ((result '()))
    (for-each
      (lambda (br)
        (for-each
          (lambda (g1)
            (for-each
              (lambda (b1)
                (let ((p1 (/ b1 g1))
                      (pr (/ br gr)))
                  (call-with-current-continuation
                    (lambda (break)
                      (for-each
                        (lambda (k)
                          (if verbose
                            (begin
                              (display
                                (format "(pr, n)=(~a, ~a), "
                                        pr n))
                              (display
                                (format "p1=~a, k=~a: ~a~a"
                                        k p1
                                        (equation-string k
                                          p1 pr n)
                                        (if (in-bound k p1 pr n)
                                          " (OK)"
                                          "")))
                              (newline)))
                          (if (not (in-bound k p1 pr n))
```

```
                              (break #f)))
                          (coprimes (lcm g1 gr)))
                        (if verbose
                          (begin
                            (display
                              (format "(pr, n)=(~a, ~a), p1=~a: "
                                      pr n p1))
                            (display "*** All solutions are ")
                            (display "algebraic ***")
                            (newline)))
                        (set! ; Found triplet, store it
                          result
                          (cons (list p1 pr n) result))))))
               (coprimes g1)))
           g1s))
       (coprimes gr))
    result))
```

As found in section 4.1, we need only consider $\gamma_1 \leq \frac{4\gamma_r}{2n-\gamma_r}$ to find all tuples for which there may be algebraic solutions. Note $\frac{4\gamma_r}{2n-\gamma_r}$ is 8, 12 resp. 6 for $(\gamma_r, n) = (4, 3)$, $(6, 4)$ resp. $(6, 5)$.

```
(show (calculate 4 3 (iota 9 1) #f))
(show (calculate 6 4 (iota 13 1) #f))
(show (calculate 6 5 (iota 7 1) #f))
```

This does pretty much the same for the values of γ_r found in section 4.2. Note that only multiples of 3 are considered.

```
(show (calculate 6 3 (iota 39/3 3 3) #f))
```

Finally, we calculate the equivalent results for the values of γ_r found in section 4.3.

```
(show (calculate 10 3 '(2) #f))
(show (calculate 10 3 (iota 165/5 5 5) #f))
```

The above program yields the following results:

```
((1/6 5/6 4) (5/6 1/6 4))
((1/6 5/6 3) (1/3 5/6 3) (5/6 1/6 3) (2/3 1/6 3))
```

Bibliography

[Beu05] Frits Beukers. Gauss' hypergeometric function. Available online at http://www.math.uu.nl/people/beukers/MRIcourse93.pdf, 2005. Lecture notes for MRI masterclass taught in 1993.

[Beu07] Frits Beukers. Algebraicity of hypergeometric functions. (Unpublished) sheets for lecture delivered at the Vrije Universiteit Amsterdam, June 2007.

[BH89] Frits Beukers and Gert Heckman. Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Inventiones Mathematica*, 95(2):325–354, 6 1989.